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VORTICAL MOMENTUM OF BOUNDED IDEAL INCOMPRESSIBLE FLUID FLOWS

V. A. Vladimirov

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1. The true momentum

$$\mathbf{I} \equiv \int \mathbf{v} dV$$

exists in three-dimensional homogeneous incompressible fluid flows filling the whole space and at rest at infinity only when the velocity field $\mathbf{v}(\mathbf{r}, t)$ satisfies the conditions [1]

$$r^3 |\mathbf{v}(\mathbf{r}, t)| \to 0 \quad \text{as} \quad r \equiv |\mathbf{r}| \to \infty, \tag{1.1}$$

which excludes the important cases of flows possessing source and dipole asymptotics. If (1.1) is satisfied, then $\mathbf{I} = 0$.

Indeed

$$\int v_i dV = \int \frac{\partial}{\partial x_k} (x_i v_k) \, dV = \int x_i v_k dS_k. \tag{1.2}$$

The continuity equation and the rule of summation over repeated indices are used, and x_k are Cartesian coordinates.

The last integral in (1.2) is taken over an infinitely remote surface. It equals zero because of (1.1) so that $\mathbf{I} = 0$. Therefore, the true momentum either does not exist for the flows under consideration, or is zero.

For this reason, the so-called "vortical" momentum of the flow is introduced

$$\mathbf{P} \equiv \frac{1}{2} \int \mathbf{r} \times \boldsymbol{\omega} dV, \quad \boldsymbol{\omega} \equiv \operatorname{rot} \mathbf{v}. \tag{1.3}$$

This quantity was defined [2] only for fluid flows filling all space. It possesses the following properties:

a) It exists if $r^4 |\boldsymbol{\omega}(\mathbf{r}, t)| \to 0$ as $r \to \infty$; this requirement is less constraining than (1.1) since it imposes a constraint on the behavior of the vortex field and not on the velocity at infinity;

b) it possesses the dimensionality of a momentum;

c) it is independent of the selection of the origin since $\int \boldsymbol{\omega} dV = 0$ in the case under consideration;

d) under the effect of external volume forces $\mathbf{f}(\mathbf{r}, t)$ it varies analogously to the physical momentum

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$$\frac{d\mathbf{P}}{dt} = \int \mathbf{f}(\mathbf{r}, t)\, dV. \tag{1.4}$$

The definition (1.3) is applicable to both ideal and viscous fluid flows.

An extension of the concept of the vortical momentum to the case of the presence of some moving bounded cavities, solid or deformable bodies in the fluid was proposed in [3]. The fluid hence remains unbounded at infinity in all directions. We call the fluid boundaries in such flows interior. According to [3], the main condition which the vortical momentum should satisfy during its extension is compliance with a dynamic equation analogous to (1.4).

At the same time, fluid flows in the presence of exterior boundaries are of great interest. In many cases, the symmetry of these boundaries permits the expectation of the presence of integral flow characteristics similar to the momentum. Among such flows, for instance, are the fluid flow in a half space bounded by a plane wall, and the flow in a pipe. It can be shown that the true momentum of the flow in a half space either exists or equals zero. For the flow in a pipe $I=0$ if the fluid is at rest at the infinitely remote endfaces of the pipe.

The question of defining the momentum concept in the presence of exterior boundaries (and particularly for the flow in a pipe) has acquired special urgency in connection with the vortex model of the loss of superfluidity by liquid helium, proposed by R. Feynman. In papers on this question, it is often assumed that the vortex perturbation is generated near the wall of a capillary, and the conditions for this generation are determined by the energy and momentum of the vortex perturbation [4, 5]. The question of the possibility of different definitions of the momentum in this case is posed in [6], where the hypothesis is expressed that the vortex rings in a superfluid fluid should be hollow. Hence, it is important to determine the magnitude of the moment for flows which have both exterior, at rest, and also interior moving boundaries. The analysis of the momentum concept is also of independent interest to hydrodynamicists.

An extension of the concept of the vortex momentum to the case of fluid flows with exterior boundaries is examined in this paper. Caverns (cavities) and solid or deformable bodies, i.e., interior boundaries not abutting the exterior boundaries, can hence be present in the fluid.

The main requirements which the momentum to be determined should satisfy are the following:

a) It should exist for a sufficiently broad class of flows and their exterior boundaries Γ, particularly for flows with boundaries going to infinity;

b) it should have the dimensionality of a momentum;

c) it should satisfy a dynamic equation which is an extension of (1.4);

d) upon removal of the outer boundaries from the body and the domain of concentrated vorticity, the quantity to be determined and the corresponding dynamical equation should go over into the known expression for the vortical momentum of a fluid unbounded at infinity [2, 3];

e) upon consideration of the motion of a body in a fluid without vortices, the momentum to be determined should agree with the known apparent momentum;

f) in the case of the already mentioned symmetry of the outer boundaries, appropriate components of the momentum to be determined should satisfy (1.4) and, in particular, should be conserved for $\mathbf{f}=0$;

g) we do not impose the requirement of independence of the quantity to be determined from selection of the origin.

Requirements d) and e) mean that the momentum to be determined should be a "local" flow characteristic in some sense. The "locality" denotes the slightness of the influence here of the removed exterior boundaries on the quantity to be determined.

In addition, an estimate of the conservation of the quantity to be determined for a typical experimental situation is presented in this paper.

At the end the extension of the concept of vortical moment of momentum [7, 3] to the flow under consideration, and also the method of extending the vortical momentum concept to the case of bounded ideal incompressible flows of an inhomogeneous (density $\rho \neq$ const) fluid, are mentioned.

2. The vortical momentum is defined in [3] for bounded bodies or cavities (interior boundaries) in a fluid

$$\mathbf{P}=\frac{1}{2}\left\{\int \mathbf{r}\times\boldsymbol{\omega}\,dV+\int_{\partial K}\mathbf{r}\times(\mathbf{n}\times\mathbf{v})\,dS\right\}. \tag{2.1}$$

For simplicity, we shall speak about one body occupying a bounded domain K. The surface integral in (2) is taken over the boundary of the body ∂K, and **n** is the exterior normal to ∂K. The first integral in (2.1) is taken over the whole fluid volume. Constraints are hence imposed on the behavior of the fields **v**(**r**, t) and **ω**(**r**, t):

$$\text{when } r\equiv|\mathbf{r}|\to\infty \quad r|\mathbf{v}(\mathbf{r},t)|\to 0,\ r^4|\boldsymbol{\omega}(\mathbf{r},t)|\to 0, \tag{2.2}$$

which permits considering both the change in the volume of the body K and its translational motion (source and dipole asymptotic). The constraints presented are not contradictory since the velocity field at large distances can be a potential field.

It is natural to consider a direct extension of (2.1) to the case of the presence of exterior boundaries Γ, for which the surface integration is extended over these boundaries as well:

$$\mathbf{P}_0=\frac{1}{2}\left\{\int \mathbf{r}\times\boldsymbol{\omega}\,dV+\int_{\partial K+\Gamma}\mathbf{r}\times(\mathbf{n}\times\mathbf{v})\,dS\right\}. \tag{2.3}$$

This definition takes account of the interior and exterior boundaries in a symmetric manner. It can be said that it is either identically the true momentum **I** or does not exist.

Indeed, in cases of the existence of **I** and $\mathbf{P}_0$ it can be shown by a simple transformation that $\mathbf{I}=\mathbf{P}_0$. Among such cases are bounded fluid flows, as well as cases of the boundaries Γ going to infinity if (1.1) is satisfied.

When (1.1) is not satisfied, then neither **I** nor $\mathbf{P}_0$ exist.

Therefore, the definition (2.3) yields nothing new compared to the true momentum **I**, and the discussion of the properties of $\mathbf{P}_0$ reduces to an analysis of the quantity **I**.

About **I** it can be said that even for such flows for which the true momentum exists, it does not satisfy the requirements listed in Sec. 1, and often turns out to be quite empty.

Thus, for flows in a bounded vessel at rest without a body K, we have $\mathbf{I}\equiv 0$ from (1.2) outside of the dependence on the size of the vessel and the acting forces. Hence **I** and $\mathbf{P}_0$ cannot satisfy a dynamical equation of the type (1.4) in this example, and are not "local" characteristics. This latter is manifest in the definition (2.3) in that the integral over Γ always remains finite, as though the vessel walls are far from the body and the domains of concentrated vorticity are not at the vessel walls.

The quantity **I** also does not satisfy requirements e) and f) from Sec. 1.

The listed disadvantages of the quantities **I** and $\mathbf{P}_0$ are associated with the fact that the strong effect of the outer boundaries Γ is taken into account.

Definition (2.1) is free of all these disadvantages if ∂K therein is understood only as the interior fluid boundary. The following assertion is valid for (2.1).

In three-dimensional flows of a homogeneous, ideal, incompressible fluid with any outer boundary Γ at rest, the vortical momentum defined in conformity to (2.1), where ∂K is understood to be the inner boundaries, will satisfy the dynamical equation

$$\frac{d\mathbf{P}}{dt}=\int_{\partial K} p\,\mathbf{n}\,dS+\int \mathbf{f}\,dV-\frac{1}{2}\int_{\Gamma}[v^2\mathbf{n}+\mathbf{r}\times\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{n})+\mathbf{r}\times(\mathbf{n}\times\mathbf{f})]\,dS, \tag{2.4}$$

where $v^2=\mathbf{v}\cdot\mathbf{v}$ and p is the pressure.

If the boundaries Γ are removed to infinity, then the constraints (2.2), corresponding to the conditions for the existence of the integrals in (2.4), are imposed on (2.2). Smoothness conditions needed for the proof are also imposed on these fields and on Γ and ∂K. Thus, the field **v** should be twice continuously differentiable. Analogous constraints are imposed on the field **f**. The interior moving boundaries ∂K are assumed not to abut on the outer boundaries Γ at rest.

Equation (2.4) is obtained quite simply for the case of no body K in the fluid, hence the integral over ∂K vanishes in (2.4). The proof of (2.4) is complicated when K is present, hence we present its basis steps here.

Let us rewrite (2.1) in tensor form

$$2P_i = \int \varepsilon_{ikl} x_k \omega_l dV + \int_{\partial K} \varepsilon_{ikl}\varepsilon_{lmn} x_k v_n dS_m,$$

where ε_{ikl} is the unit absolutely antisymmetric tensor of the third rank, and dS_i is the component of the vector surface element directed into the fluid. The rule of summation over repeated subscripts is used throughout.

Since they bound the fluid volume, the surfaces ∂K and Γ consist of the very same fluid particles all the time, i.e., are fluid surfaces [8].

Using the known rule of differentiation with respect to the fluid volumes and surfaces [7], we obtain

$$2\frac{dP_i}{dt} = \int \varepsilon_{ikl}\left(v_k\omega_l + x_k\frac{d\omega_l}{dt}\right)dV + \int_{\partial K} \varepsilon_{ikl}\varepsilon_{lmn}\left(v_k v_n + x_k\frac{dv_n}{dt}\right)dS_m - \int_{\partial K} \varepsilon_{ikl}\varepsilon_{lmn} x_k v_n \frac{\partial v_a}{\partial x_m} dS_a. \tag{2.5}$$

Using the equations of motion of an ideal fluid in the form

$$\frac{dv_n}{dt} = -\frac{\partial p}{\partial x_n} + f_n, \quad \frac{\partial v_a}{\partial x_a} = 0,$$

$$\frac{d\omega_l}{dt} = -\varepsilon_{lmn}\frac{\partial v_a}{\partial x_m}\frac{\partial v_n}{\partial x_a} + \varepsilon_{lmn}\frac{\partial f_n}{\partial x_m},$$

we convert (2.5) to the form

$$2\frac{dP_i}{dt} = 2\int_{\partial K} p\,dS_i - \int_{\Gamma} v^2 dS_i + 2\int f_i dV + \int_{\Gamma} \varepsilon_{ikl}\varepsilon_{lmn} x_k v_n \frac{\partial v_a}{\partial x_m} dS_a - \int_{\Gamma} \varepsilon_{ikl}\varepsilon_{lmn} x_k f_n dS_m. \tag{2.6}$$

The subsequent problem is to transform the integral

$$\int_{\Gamma} \varepsilon_{ikl}\varepsilon_{lmn} x_k v_n \frac{\partial v_a}{\partial x_m} dS_a. \tag{2.7}$$

We use the following method to do this. Let us continue the field $\mathbf{v}(\mathbf{r}, t)$ in an arbitrarily sufficiently smooth manner on K, after which the possibility of converting the integral over Γ to volume integrals over the whole domain bounded by the exterior boundaries is manifest. Here, in general, $\omega = \operatorname{rot} \mathbf{v} \neq 0$, Q div $\mathbf{v} \neq 0$ on K. Converting (2.7) by using the identity

$$\varepsilon_{lmn}\frac{\partial v_k}{\partial x_m}\frac{\partial v_n}{\partial x_k} = -\omega_m\frac{\partial v_l}{\partial x_m} + \omega_l Q, \tag{2.8}$$

we obtain

$$\int_{\Gamma} \varepsilon_{ikl}\varepsilon_{lmn} x_k v_n \frac{\partial v_a}{\partial x_m} dS_a = -\int_{\Gamma} \varepsilon_{ikl} x_k v_l \omega_m n_m dS,$$

after which we obtain (2.4) from (2.6). The identity (2.8) is obtained by using the application of the operation rot (curl) to the vector equality

$$v_a\frac{\partial v_n}{\partial x_a} = \frac{\partial}{\partial x_n}\left(\frac{v^2}{2}\right) - \varepsilon_{npq} v_p \omega_q.$$

The proof presented for the relationship (2.4) is greatly analogous to the proof presented in [3].

If we introduce the momentum $\mathbf{P}_1$ of the body K, then we will have for the sum $\mathbf{P}_1 + \mathbf{P}$

$$\frac{d}{dt}(\mathbf{P}_1 + \mathbf{P}) = \mathbf{F} - \frac{1}{2}\int_{\Gamma}\{v^2\mathbf{n} + \mathbf{r}\times\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{n}) + \mathbf{r}\times(\mathbf{n}\times\mathbf{f})\}\,dS, \tag{2.9}$$

where $\mathbf{F}$ is the total exterior force acting on the body K and the fluid.

The principal difference between (2.4), (2.9), and the corresponding equations for the true momentum is the absence of $\int\limits_{\Gamma} p\mathbf{n}dS$, expressing the force effect of the exterior boundaries on the fluid, from the first terms. A sufficiently complicated integral over Γ satisfying the requirement formulated in Sec. 1 enters instead: Upon removal of the boundaries Γ from the body K and the region of concentrated $\boldsymbol{\omega}$ and $\mathbf{f}$, this integral tends to zero so that (2.4) and (2.9) reduce to equations for the unbounded fluid [3]:

$$\frac{dP_i}{dt} = \int\limits_{\partial K} pn_i dS + \int f_i dV, \quad \frac{d}{dt}(\mathbf{P}_1 + \mathbf{P}) = \mathbf{F}. \tag{2.10}$$

3. If $\mathbf{f}\times\mathbf{n}=0$ and $\boldsymbol{\omega}\cdot\mathbf{n}=0$ on Γ, then (2.4) and (2.9) are simplified substantially:

$$\frac{d\mathbf{P}}{dt} = \int\limits_{\partial K} p\mathbf{n}dS + \int \mathbf{f}dV - \frac{1}{2}\int\limits_{\Gamma} v^2\mathbf{n}dS. \tag{3.1}$$

$$\frac{d}{dt}(\mathbf{P}_1 + \mathbf{P}) = \mathbf{F} - \frac{1}{2}\int\limits_{\Gamma} v^2\mathbf{n}dS.$$

It is seen that for flows above a plane (3.1) agrees with (2.10) for momentum components parallel to the plane, in conformity with the requirement for symmetric exterior boundaries formulated in Sec. 1. The same can be said about the axial component of the momentum in a pipe and about the component of $\mathbf{P}$ tangent to the planes in the gap between the planes.

The case with $\boldsymbol{\omega}\cdot\mathbf{n}\neq 0$ on Γ is more complicated. It is especially important in connection with models of the loss of superfluidity by helium [5]. In this case, the vortex lines are already not closed outside of Γ and the quantity $\mathbf{P}$ (2.1) turns out to be dependent on the selection of the selection of the origin.

For simplicity, let us consider a flow without a body K. Then the vortical momentum will change by the quantity

$$\frac{1}{2}\mathbf{R}\times\int \boldsymbol{\omega}dV \tag{3.2}$$

when the origin is shifted by the vector $\mathbf{R}$. In the case of constant (3.2) this quantity is negligible for the dynamical analysis. However, this is not so in the general case since

$$\frac{d}{dt}\int \boldsymbol{\omega}dV = \int\limits_{\Gamma} \mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{n})\,dS \neq 0. \tag{3.3}$$

The proof of the assertion (3.3) does not exist in general form. Moreover, examples can be constructed in which (3.3) is not satisfied. The physical mechanism specifying the nonconservation of $\int \boldsymbol{\omega}dV$ is tension and compression of the vortex lines, hence (3.3) is valid for general three-dimensional flows. Construction of confirming examples for the boundaries Γ of a general form cause no difficulties. To prove (3.3) in the case of flow over a plane, let us consider the following example. Its crux is to construct a flow for which the conservation of the quantity $\int \boldsymbol{\omega}dV$ is incompatible with law of conservation of the vortical momentum (2.1) proved earlier.

Two coaxial vortex semirings with thin cores move over a plane so that the ends of these semirings rest in the plane Γ perpendicularly to it. Upon adding this flow to the flow in all space by the method of constructing a "reflection," we obtain the problem of interaction between two coaxial thin vortex rings.

In this case the considered conservation law for the vortical momentum takes the form

$$\frac{d}{dt}\left\{l_1^2\Gamma_1\left[1+O\left(\frac{a_1}{l_1}\right)\right] + l_2^2\Gamma_2\left[1+O\left(\frac{a_2}{l_2}\right)\right]\right\} = 0,$$

where l_α, Γ_α, and a_α are, respectively, the radius, circulation, and radius of the core section of the α-th half-ring, $a_\alpha/l_\alpha \ll 1$ ($\alpha = 1, 2$).

This law is incompatible with the constancy of $\int \boldsymbol{\omega}dV$, which is written in the form

$$\frac{d}{dt}\left\{l_1\Gamma_1\left[1+O\left(\frac{a_1}{l_1}\right)\right] + l_2\Gamma_2\left[1+O\left(\frac{a_2}{l_2}\right)\right]\right\} = 0.$$

To prove this assertion it is necessary to involve the conservation law for the core volume (not summed over α)

$$\frac{d}{dt}\left\{a_\alpha^2 l_\alpha\left[1+O\left(\frac{a_\alpha}{l_\alpha}\right)\right]\right\} = 0 \quad (\alpha = 1, 2),$$

as well as the fact that a_α/l_α can be selected arbitrarily less than any previously assigned number.

Therefore, because of (3.3) the right side of (2.4) has nonzero tangential components for the case of flow above a plane if the origin is chosen not on the plane. If the origin is selected on Γ and $\mathbf{n}\times\mathbf{f}=0$ on Γ, then we again have (2.10) for the components of **P** tangent to Γ. This follows from $\mathbf{r}\times\mathbf{v}\|\mathbf{n}$ on Γ.

Only in special cases can (2.10) be obtained for components tangent to the planes for flows in slots between parallel planes with $\omega\cdot\mathbf{n}\neq 0$ on Γ. Among these are plane flows and flows in which all the vortex lines can be terminated only on one of the planes. In the latter case (2.10) is obtained by selecting the origin on that plane to which the vortex lines abut.

The existence of such special cases is also possible for the flow in a pipe. However, it is clear that such cases are exceptional for pipes and slots, although their classification is a difficult problem.

It should be noted that flows with $\omega\cdot\mathbf{n}\neq 0$ on Γ are flows with a singularity on the boundary. Thus, for the flow over a plane, the known method of adding a flow to the flow in all space by constructing the "reflection" results in breaks in ω on Γ. As is known, these latter are associated with singularities in the velocity field [7]. Hence, in order for the presented proof of (2.4) to be rigorous for $\omega\cdot\mathbf{n}\neq 0$ and Γ, it is necessary either to prove the necessary differentiability of the velocity field, or to impose the condition $\omega\times\mathbf{n}=0$ on Γ, which is dynamical for the flows over a plane and in a slot. This means that the validity for all t follows from its compliance at t=0, which is seen at once after the construction of the "reflection." It is true that here and throughout the assumption has been used about the conservation of the necessary smoothness of the flow with time. This question is complex and unsolved [9].

The results obtained for flows in a pipe, slot, and half-space can be summarized as follows:

a) For all these flows, the requirement $\omega\cdot\mathbf{n}=0$ on Γ, which is evidently dynamical, is the sufficient condition for the corresponding vortical momentum components to vary analogously to the true momentum (1.4) and (2.10);

b) for $\omega\cdot\mathbf{n}\neq 0$ on Γ the same result can be obtained for the flow over a plane by selecting the origin thereon;

c) for flows in a slot and pipe (2.10) for vortical momentum components parallel to the walls can be obtained only for flows of a special kind. In general, it is impossible to do this for the momentum (2.1).

4. Let us now examine the case when the boundaries are removed from the body K and the concentration region ω, f=0. This corresponds to the often encountered experimental situations, for instance, to the motion of a vortex ring in a room. In this case (3.1) affords the possibility of estimating the accuracy of conserving the vortex momentum. This estimate is simultaneously a verification of the customary method of describing the vortex momentum of real bounded flows by using the model of a fluid unbounded at infinity.

By using the expression for the principal term of the asymptotic of the velocity field of unbounded incompressible fluid flows [7]

$$v_i \approx \frac{P_k}{4\pi}\frac{\partial^2}{\partial x_i \partial x_k}\left(\frac{1}{r}\right) \quad \text{as} \quad r\to\infty$$

and (3.1), we obtain the estimate

$$\left|\frac{d\mathbf{P}}{dt}\right| \approx \frac{A}{8\pi}\cdot\frac{|\mathbf{P}|^2}{R^4},$$

where R is the characteristic distance to the outer walls Γ; A is a constant of the order of one, dependent on the geometry of the walls Γ.

We obtain the estimate of the change in momentum because of the influence of external walls for a vortex ring in a vessel

$$\frac{|\Delta\mathbf{P}|}{|\mathbf{P}|} \approx \frac{A}{8\pi}\frac{Ut}{R}\left(\frac{l}{R}\right)^3,$$

where U and l are the ring velocity and size. For $l/R\sim 0.1$ and $R\sim Ut$ we obtain $|\Delta\mathbf{P}|/|\mathbf{P}|\sim 10^{-4}$, i.e., conservation of the vortex momentum with high accuracy.

5. According to the method proposed in this paper, the extension of the concept of the vortical momentum and the ideas of [3], the vortical momentum can also be introduced for bounded flows of an ideal incompressible inhomogeneous (density $\rho\neq$ const) fluid.

Methods analogous to that considered also permit introduction of the concept of the vortical moment of momentum [7, 3] of bounded fluids. Thus, an equation of the type (2.4) can be obtained for the quantity

$$\mathbf{M} = \frac{1}{3}\left\{\int \mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})\,dV + \int_{\partial K} \mathbf{r}\times[\mathbf{r}\times(\mathbf{n}\times\mathbf{v})]\,dS\right\}.$$

This definition is also extended to the case of inhomogeneous, incompressible fluid flows.

The extension of the results of this paper to the case of viscous fluid flows bounded by external walls at rest is also possible although difficulties with vorticity diffusing from these walls occur here. It is possible to obtain an equation of type (2.10) for flows in a slot, pipe, and half-space if this vorticity is assumed localized near the walls Γ and it is not included in the definition of the momentum (2.1) since otherwise the integrals in (2.1) cannot exist.

The following can be said with respect to the application of the concept of the vortical momentum (1.3) to obtaining the criterion for the loss of superfluidity of liquid helium. The concept of the true momentum, which does not at all agree with the vortical momentum for the flow in a capillary or near its exit, was used in [10] to obtain the criterion for the loss of superfluidity. Hence, the use of the vortical momentum in this case is a hypothesis based on dimensional analysis, but no more. The situation is aggravated by the fact that, according to the results of Sec. 3, the vortical momentum has no physical meaning for the case $\omega \cdot \mathbf{n} \neq 0$ on Γ.

It hence follows that the application itself of the concept of the vortical momentum to the problem of the loss of superfluidity requires critical analysis and, possibly, reexamination.

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